Problems Optimization

This material corresponds roughly to sections 14.7 and 14.8 in the book.

Problem 1. Find the shortest distance from the plane $3x - 2y - z = 3$ to the origin using Lagrange multipliers.

We must minimize $s(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to the condition $g(x, y, z) = 3x - 2y - z - 3 = 0$. To make things easier, notice that we can actually minimize $f(x, y, z) = s^2(x, y, z) = x^2 + y^2 + z^2$, since the distance will be a minimum if and only if its square is a minimum.

Given that

$$\begin{cases} \nabla f = (2x, 2y, 2z) \\ \nabla g = (3, -2, -1) \end{cases} \tag{1}$$

we need to solve the system of equations

$$\begin{cases} 2x = 3\lambda \\ 2y = -2\lambda \\ 2z = -\lambda \\ 3x - 2y - z = 3 \end{cases} \tag{2}$$

Notice that the first three equations give us

$$x = \frac{3\lambda}{2}, y = -\lambda, z = -\frac{\lambda}{2} \tag{3}$$

and so if we substitute this information into the last equation we find that

$$\frac{9\lambda}{2} + 2\lambda + \frac{\lambda}{2} = 3 \tag{4}$$

which means that

$$\lambda = \frac{3}{7} \tag{5}$$

For this value of λ we have

$$\boxed{x = \frac{9}{14}, y = -\frac{3}{7}, z = -\frac{3}{14}} \tag{6}$$

and the distance is then

$$\boxed{s = \sqrt{\left(\frac{9}{14}\right)^2 + \left(-\frac{3}{7}\right)^2 + \left(-\frac{3}{14}\right)^2} = \frac{3}{\sqrt{14}}} \tag{7}$$

Problem 2. A rectangular box, open at the top, is to hold 256 cubic centimeters of cat food. Find the dimensions for which the surface area (bottom and four sides) is minimized.

Call the sides x, y, z. We need to minimize

$$S = xy + 2xz + 2yz \tag{8}$$

subject to the condition

$$V = xyz = 256 \tag{9}$$

Since

$$\begin{cases} \nabla S = (y + 2x, x + 2z, 2x + 2y) \\ \nabla V = (yz, xz, xy) \end{cases} \tag{10}$$

So we must solve the system

$$\begin{cases} y + 2x = \lambda yz \\ x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz = 256 \end{cases} \tag{11}$$

Multiply the first equation by x, the second by y, and the third by z in order to obtain the new system

$$\begin{cases} xy + 2x^2 = \lambda xyz \\ xy + 2yz = \lambda xyz \\ 2xz + 2yz = \lambda xyz \\ xyz = 256 \end{cases} \tag{12}$$

The second and third equations imply that

$$xy + 2yz = 2xz + 2yz \implies xy = 2xz \implies \boxed{y = 2z} \tag{13}$$

Notice that we implicitly omitted the case $x = 0$, since it is not useful geometrically. We substitute this back into the system of equations to obtain

$$\begin{cases} 2xz + 2x^2 = 2\lambda xz^2 \\ 2xz + 4z^2 = 2\lambda xz^2 \\ 2xz + 4z^2 = 2\lambda xz^2 \\ 2xz^2 = 256 \end{cases} \tag{14}$$

Now we equate the first and second equations to obtain

$$2xz + 2x^2 = 2xz + 4z^2 \implies x^2 = 2z^2 \implies \boxed{x = \sqrt{2}z} \tag{15}$$

Strictly speaking, $x = -\sqrt{2}z$ is a possibility, but we exclude it since the sides of a rectangle must be of positive health. Substituting into the last equation we need to solve

$$2\sqrt{2}z^3 = 256 \implies \boxed{z = 4\sqrt[6]{2}} \tag{16}$$

And so the dimensions of the rectangle are

$$\boxed{x = 4\sqrt{2}\sqrt[6]{2}, y = 8\sqrt[6]{2}, z = 4\sqrt[6]{2}} \tag{17}$$

Problem 3. Suppose that the output of a manufacturing firm is a quantity Q of product which is a function K of capital equipment or investment and the amount of labor L used. For example, the Cobb-Douglas production function is $Q(K, L) = AK^\alpha L^{1-\alpha}$, where A, α are positive constants and $\alpha < 1$. This is sometimes a simple model for the national economy. If the price of labor is p, the price of capital is q, and the firm can spend no more than B dollars, find the amount of capital and labor which maximizes the output Q. Solve this problem using the Lagrange multiplier method.

We want to maximize

$$Q = AK^\alpha L^{1-\alpha} \tag{18}$$

subject to the condition

$$qK + pL = B \tag{19}$$

In this case we need to solve

$$\begin{cases} \alpha AK^{\alpha-1}L^{1-\alpha} = \lambda q \\ (1-\alpha)AK^\alpha L^{-\alpha} = \lambda p \\ qK + pL = B \end{cases} \tag{20}$$

Multiply the first equation by p and the second by q to obtain the system

$$\begin{cases} \alpha pAK^{\alpha-1}L^{1-\alpha} = \lambda pq \\ (1-\alpha)qAK^\alpha L^{-\alpha} = \lambda qp \\ qK + pL = B \end{cases} \tag{21}$$

Setting the first two equations equal to each other to obtain

$$\alpha pAK^{\alpha-1}L^{1-\alpha} = (1-\alpha)qAK^\alpha L^{-\alpha} \implies \alpha pK^{-1}L = (1-\alpha)q \implies \boxed{L = \frac{(1-\alpha)q}{\alpha p}K} \tag{22}$$

Substituting into the last equation we find that

$$qK + \frac{(1-\alpha)q}{p}K = B \tag{23}$$

which we can rewrite as

$$pqK + (1-\alpha)qK = pB \tag{24}$$

and therefore

$$\boxed{K = \frac{pB}{pq + (1-\alpha)q}} \tag{25}$$

The corresponding value for L is

$$\boxed{L = \frac{(1-\alpha)q}{\alpha p}\frac{pB}{pq+(1-\alpha)q} = \frac{(1-\alpha)B}{\alpha[p+(1-\alpha)]}} \tag{26}$$

Problem 4. Find the constants a, b for which $F(a,b) = \int_0^\pi (\sin x - (ax^2+bx))^2 dx$ is a minimum. Check that these values of a, b do correspond to a minimum.

One option to solve this problem is by expanding the integrand. However, it is easier to differentiate under the integrand sign. Namely,

$$\begin{aligned}
&\frac{\partial}{\partial a}F \\
&= \int_0^\pi \frac{\partial}{\partial a}\left[(\sin x - (ax^2+bx))^2\right] dx \\
&= \int_0^\pi -2(\sin x - (ax^2+bx))x^2 dx \\
&= -2\left[\int_0^\pi (x^2\sin x - (ax^4+bx^3))dx\right] \\
&= -2\left[\pi^2 - 4 - a\frac{\pi^5}{5} - b\frac{\pi^4}{4}\right]
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{\partial}{\partial b}F \\
&= \int_0^\pi \frac{\partial}{\partial b}\left[(\sin x - (ax^2+bx))^2\right] dx \\
&= \int_0^\pi -2(\sin x - (ax^2+bx))x dx \\
&= -2\left[\int_0^\pi (x\sin x - (ax^3+bx^2))dx\right] \\
&= -2\left[\pi - a\frac{\pi^4}{4} - b\frac{\pi^3}{3}\right]
\end{aligned}$$

The other partial derivatives are

$$\begin{cases} \frac{\partial^2 F}{\partial a^2} = 2\frac{\pi^5}{5} \\ \frac{\partial^2 F}{\partial a \partial b} = \frac{\pi^4}{2} \\ \frac{\partial^2 F}{\partial b^2} = 2\frac{\pi^3}{3} \end{cases} \tag{27}$$

The critical points solve the equation

$$\begin{cases} -2\left[\pi^2 - 4 - a\frac{\pi^5}{5} - b\frac{\pi^4}{4}\right] = 0 \\ -2\left[\pi - a\frac{\pi^4}{4} - b\frac{\pi^3}{3}\right] = 0 \end{cases} \tag{28}$$

The solution to this system is

$$\boxed{a = \frac{20}{\pi^3} - \frac{320}{\pi^5}, b = \frac{240}{\pi^4} - \frac{12}{\pi^2}} \tag{29}$$

This corresponds to a minimum since the Hessian is

$$H(a,b) = \begin{pmatrix} 2\frac{\pi^5}{5} & \frac{\pi^4}{2} \\ \frac{\pi^4}{2} & 2\frac{\pi^3}{3} \end{pmatrix} \tag{30}$$

The determinant of the Hessian is

$$\det H(a,b) = \frac{4\pi^8}{15} - \frac{\pi^8}{4} = \frac{\pi^8}{60} > 0 \tag{31}$$

and the second derivative $\frac{\partial^2 F}{\partial a^2} = 2\frac{\pi^5}{5}$ is positive as well, so we do get a minimum!

Problem 5. Find the shortest distance from the origin to the curve of intersection of the surfaces $xyz = a$, $y = bx$ where $a > 0$, $b > 0$.

Again, we can minimize the function $s(x,y,z) = x^2+y^2+z^2$, subject to two constraints $g_1(x,y,z) = xyz - a = 0$ and $g_2(x,y,z) = y - bx = 0$. Since

$$\begin{cases} \nabla s = (2x, 2y, 2z) \\ \nabla g_1 = (yz, xz, xy) \\ \nabla g_2 = (-b, 1, 0) \end{cases} \tag{32}$$

we must solve the system

$$\begin{cases} 2x = \lambda yz - \mu b \\ 2y = \lambda xz + \mu \\ 2z = \lambda xy \\ xyz = a \\ y = bx \end{cases} \tag{33}$$

Notice that the third and last equation gives us

$$\boxed{z = \frac{\lambda xy}{2} = \frac{\lambda b x^2}{2}} \tag{34}$$

and substituting this and the last equation into the fourth equation gives us

$$x\,(bx)\left(\frac{\lambda b x^2}{2}\right) = a \tag{35}$$

which means that

$$\boxed{x = \sqrt[4]{\frac{2a}{b^2\lambda}}} \tag{36}$$

Therefore

$$\boxed{x=\sqrt[4]{\frac{2a}{b^2\lambda}}, y=b\sqrt[4]{\frac{2a}{b^2\lambda}}, z=\frac{\lambda b}{2}\sqrt{\frac{2a}{b^2\lambda}}=\sqrt{\frac{a\lambda}{2}}} \tag{37}$$

Now we just need to find the value of λ. Substitute the value of

$$\mu = 2y-\lambda xz \tag{38}$$

from the second equation into the first equation to obtain that

$$2x=\lambda yz-2yb+\lambda bxz \tag{39}$$

in other words,

$$2\sqrt[4]{\frac{2a}{b^2\lambda}}=\lambda b\sqrt[4]{\frac{2a}{b^2\lambda}}\sqrt{\frac{a\lambda}{2}}-2b\sqrt[4]{\frac{2a}{b^2\lambda}}b+\lambda b\sqrt[4]{\frac{2a}{b^2\lambda}}\sqrt{\frac{a\lambda}{2}} \tag{40}$$

This equation is equivalent to

$$2=\lambda b\sqrt{\frac{a\lambda}{2}}-2b^2+\lambda b\sqrt{\frac{a\lambda}{2}} \tag{41}$$

Or in other words

$$1+b^2=\lambda^{3/2}b\sqrt{\frac{a}{2}} \tag{42}$$

So

$$\lambda^3=\frac{(1+b^2)^2}{b^2}\frac{2}{a} \implies \boxed{\lambda=\left\{\frac{2}{a}\left(\frac{(1+b^2)^2}{b^2}\right)\right\}^{1/3}=\frac{2^{1/3}}{a^{1/3}}\frac{(1+b^2)^{2/3}}{b^{2/3}}} \tag{43}$$

Therefore

$$\boxed{\begin{array}{rclcl} x= & & \sqrt[4]{\frac{2a}{b^2\lambda}} & = & \sqrt[4]{\frac{2aa^{1/3}b^{2/3}}{b^2 2^{1/3}(1+b^2)^{2/3}}} \\ y= & b & \sqrt[4]{\frac{2a}{b^2\lambda}} & = & \sqrt[4]{\frac{2b^2aa^{1/3}b^{2/3}}{2^{1/3}(1+b^2)^{2/3}}} \\ z= & & \sqrt{\frac{a\lambda}{2}} & = & \sqrt{\frac{a}{2}\frac{2^{1/3}}{a^{1/3}}\frac{(1+b^2)^{2/3}}{b^{2/3}}} \end{array}} \tag{44}$$

So the distance is

$$\sqrt{x^2+y^2+z^2}=\sqrt{\sqrt[4]{\frac{2aa^{1/3}b^{2/3}}{b^2 2^{1/3}(1+b^2)^{2/3}}}+\sqrt[4]{\frac{2b^2aa^{1/3}b^{2/3}}{2^{1/3}(1+b^2)^{2/3}}}+\sqrt{\frac{a}{2}\frac{2^{1/3}}{a^{1/3}}\frac{(1+b^2)^{2/3}}{b^{2/3}}}} \tag{45}$$

Problem 6. Prove that the shortest distance from the point (a, b, c) to the plane $Ax + By + Cz + D = 0$ is

$$\left|\frac{Aa+Bb+Cc+D}{\sqrt{A^2+B^2+C^2}}\right| \tag{46}$$

Again, we must minimize $f(x, y, z) = (x-a)^2+(y-b)^2+(z-c)^2$ subject to $g(x, y, z) =$

$Ax + By + Cz + D = 0$. This gives us the system

$$\begin{cases} 2(x-a) = \lambda A \\ 2(y-b) = \lambda B \\ 2(z-c) = \lambda C \\ Ax + By + Cz = -D \end{cases} \tag{47}$$

The first three equations gives us

$$\boxed{x = \frac{\lambda A}{2} + a, y = \frac{\lambda B}{2} + b, z = \frac{\lambda C}{2} + c} \tag{48}$$

and substituting into the last equation we have

$$\frac{\lambda A^2}{2} + Aa + \frac{\lambda B^2}{2} + Bb + \frac{\lambda C^2}{2} + Cc = -D \tag{49}$$

which means that

$$\boxed{\lambda = -2\frac{Aa + Bb + Cc + D}{A^2 + B^2 + C^2}} \tag{50}$$

and so the distance is

$$\begin{aligned} &\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \\ =&\sqrt{\frac{\lambda^2 A^2}{4} + \frac{\lambda^2 B^2}{4} + \frac{\lambda^2 C^2}{4}} \\ =&\frac{|\lambda|}{2}\sqrt{A^2 + B^2 + C^2} \\ =&\left(\frac{|Aa + Bb + Cc + D|}{A^2 + B^2 + C^2}\right)\sqrt{A^2 + B^2 + C^2} \\ =&\frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$